

Irreversible Behavior of Interacting Systems. I. The Approach to Equilibrium

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Received October 11, 1973

The Kac ring model is used to test the validity of some conjectures about irreversibility. If the whole system is regarded as the universe, then it is demonstrated that all clocks (subsystems) run in the same direction during those times when the universe is not in equilibrium. In addition, mathematical techniques are introduced by means of which the appropriate order parameter for large, finite Kac ring models can be evaluated asymptotically. It is shown that the relaxation of this order parameter to its equilibrium value of zero is not exponential.

KEY WORDS: Nonequilibrium system; irreversibility; arrow of time; clocks; Poincaré cycle; Kac ring model; order parameter; relaxation time; steepest descent.

1. INTRODUCTION

In looking over the literature on the subject of irreversibility, one is immediately struck by the fact that most of the ideas presented are at once self-evident and undemonstrable.⁽¹⁻³⁾ This apparent contradiction is perhaps not too surprising in a subject where the statements of the questions themselves are so ill-defined as to virtually preclude the creation of a conceptual framework (i.e., model) within which calculations can be made. We need only point out that calculations of transport coefficients, relaxation times, and other

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nonequilibrium properties of physical systems are based on, rather than substantiate, our intuitive ideas of what a system consisting of a large number of interacting particles should do.⁽⁴⁻⁶⁾ Indeed, it is necessary to introduce a hypothesis concerning the behavior of such a system in a statistical sense (Boltzmann's Stosszahlansatz or the notion of molecular chaos) in order to first write down the equation governing the motion of macroscopic parameters (Boltzmann transport equation).^(4,6) This is not to deprecate those results which are obtained through great effort and ingenuity and which compare favorably with experiment. They are evidence that our underlying assumptions might just be correct. But our understanding of why our intuitive reasoning is right is not greatly enhanced thereby.

It seems to us that the theoretical challenge is, at the very least, to present a model whose (rigorously derived) behavior is in accord (or discord) with established belief. In this and the following paper we have resurrected a very simple many-particle system which was originally proposed to illustrate the approach to equilibrium and a derivation of a simplified Boltzmann equation from the dynamical equations governing its motion. This system is referred to in the literature as the Kac ring model⁽⁷⁾ and, except for a recent generalization of the dynamics,⁽⁸⁾ has fallen into disuse for the obvious reason that its solution was given completely at the time of its inception. This paper is concerned with a derivation of the behavior of subsystems of the original Kac model, which will provide the basis for a discussion of the arrow of time. The next paper treats the fluctuations of both the original Kac model and its subsystems.

The main difficulty associated with discussions of the arrow of time is the use of language which presupposes its very existence. Words such as develop, evolve, change, and move, although widely favored, should be avoided if we are not to fall into the trap of assuming that which we set out to prove. The phrase "approach to equilibrium," while seemingly innocuous, appears to indicate a preferred direction on the time axis. Time should, in fact, be considered purely as a coordinate for observable phenomena with no direction implied or expressed. If the arrow of time is to be established, it must result from the functional form of time-dependent phenomena and not simply as a consequence of the way we have chosen to draw the axis.

Before specializing to the model we will review some general features of the real universe. Many observable parameters of the universe are not constants in time. If these parameters are associated with nearly isolated systems and change monotonically with time, then the associated systems serve as clocks. (The word "clock" here is used to denote something which indicates the direction of the time axis and does not necessarily refer to a counter which marks off equal intervals on that axis.) The kind of parameter we have in mind for this discussion is exemplified by Boltzmann's H -functional. It is

observed that the slope of this functional as a function of time has the same sign for all systems. The negative or “downhill” side points in the direction of the “future”; hence the phrase “arrow of time.”

It is important to distinguish at least two aspects of the arrow of time question. The first aspect is that described above, popularly stated, “Why do all clocks go in the same direction?” The other aspect is characterized by the question, “Why does an intelligent being have knowledge (memory) of events on only one side of his projection onto the time axis and why is this side associated with greater order (larger H -functional)?” We are only concerned in these two papers with the first question. The answer to the second question lies outside the scope of present-day physics.

2. GENERAL PROPERTIES OF KAC MODELS

We will begin with a brief review of the Kac model, following the notation of Dresden⁽⁷⁾ for the most part. The Kac model consists of a ring of n positions on which m scatterers are placed randomly. There is a particle at each position and the particle can be in one of two possible states, e.g. white or black. The state η of a particle is given by

$$\begin{aligned} \eta_p(t) &= +1 && \text{if the particle at position } p, \text{ time } t, \text{ is white} \\ \eta_p(t) &= -1 && \text{if the particle at position } p, \text{ time } t, \text{ is black} \end{aligned}$$

The state ϵ of the position can be expressed in a similar manner as

$$\begin{aligned} \epsilon_p &= +1 && \text{if there is no scatterer at position } p \\ \epsilon_p &= -1 && \text{if there is a scatterer at position } p \end{aligned}$$

The system dynamics is as follows: In one unit of time all particles jump to the adjacent position in the counterclockwise direction. If there is a scatterer at a particular position, the particle leaving that position changes state upon jumping to its new position. The equation of motion is

$$\eta_p(t + 1) = \epsilon_{p+1} \eta_{p+1}(t) \tag{1}$$

from which

$$\eta_p(t) = \epsilon_{p+1} \cdots \epsilon_{p+t} \eta_{p+t}(0) \tag{2}$$

The microstate of the system is $\{\eta_p(t)\}$. A macrostate can be introduced by defining the order parameter

$$\Gamma(t) \equiv (1/n) \sum_{p=1}^n \eta_p(t) \tag{3}$$

Of particular interest is the behavior of the order parameter after preparation

of the system in a highly ordered state, $\Gamma(0) = 1$. Combining Eqs. (2) and (3) yields

$$\Gamma(t) = (1/n) \sum_{p=1}^1 \epsilon_{p+1} \cdots \epsilon_{p+t} \quad (4)$$

which is the order parameter after preparation in a highly ordered state at time $t = 0$.

An ensemble average can be obtained by averaging over all possible arrangements of the m scatterers on the n positions:

$$\langle \Gamma(t) \rangle = \sum_s (-1)^s G(s) \quad (5)$$

where

$$G(s) = \binom{t}{s} \binom{n-t}{m-s} / \binom{n}{m} \quad (6)$$

Although Eq. (5) cannot be simplified further, it can be reduced to a more manageable form in the thermodynamic limit, $n \gg 1$. We first look at times when $t \ll n$ and $\mu = m/n$ is finite. The asymptotic form becomes

$$G(s) \approx \binom{t}{s} \mu^s (1 - \mu)^{t-s} \quad (7)$$

Substituting Eq. (7) into Eq. (5), we obtain

$$\langle \Gamma(t) \rangle \approx (1 - 2\mu)^t \quad (8)$$

Because we are interested in large, finite Kac models, we must not neglect the behavior of Eq. (5) for $t \approx n$. To facilitate this examination, we introduce $t' = n - t$ and $s' = m - s$. In terms of our new variables we write Eq. (5) as

$$\langle \Gamma(t) \rangle = (-1)^m \sum_{s'} (-1)^{s'} G(s') \quad (9)$$

and for $t' \ll n$ we follow the same procedure as above and obtain

$$\langle \Gamma(t) \rangle \approx (-1)^m (1 - 2\mu)^{n-t} \quad (10)$$

At all other times, t, t' finite fractions of n , there is no simple form for Eq. (5). However, the magnitude of $\langle \Gamma(t) \rangle$ for these times is of the order $n^{-1/2}$ or smaller and it seems reasonable to consider these portions negligible in comparison to the exponential portions of $\langle \Gamma(t) \rangle$. Combining Eqs. (8) and (10) yields

$$\langle \Gamma(t) \rangle \approx (1 - 2\mu)^t \pm (1 - 2\mu)^{n-t} \quad (11)$$

Expression (5) can be evaluated partially, using the method of steepest descents. This method allows us to express $\langle \Gamma \rangle$ as an expansion in powers of

t/n and $(n - t)/n$. The evaluation is carried out in the appendix. Equation (11) results when we drop all powers of t/n and $(n - t)/n$ but the zeroth. The first-order approximation is

$$\langle \Gamma(t) \rangle \approx \exp \left[\frac{\alpha^2 m t^2}{2n(n - m)} \right] \tanh^t \left\{ \frac{1}{2} \left[\ln \frac{n - m}{m} - \frac{\alpha t}{n - m} \right] \right\} \quad (12)$$

for $t \leq \frac{1}{2}n - [m(n - m)]^{1/2}$, $\alpha = 1 + [(n - t)/(n - 2m)]$; and

$$\langle \Gamma(t) \rangle \approx \exp \left[\frac{\beta^2 m (n - t)^2}{2n(n - m)} \right] \tanh^{n-t} \left\{ \frac{1}{2} \left[\ln \frac{n - m}{m} - \frac{\beta(n - t)}{n - m} \right] \right\} \quad (13)$$

for $t \geq \frac{1}{2}n + [m(n - m)]^{1/2}$, $\beta = 1 + [t/(n - 2m)]$. These equations indicate that the approach to equilibrium in a Kac model is actually somewhat steeper than the exponential decay predicted by Eq. (11). For large, dilute systems, however, the error in Eq. (11) can be considered negligible.

3. INTERACTING KAC MODEL SUBSYSTEMS

We will now expand the applicability of the Kac model to examine the relationship of weakly interacting systems. Consider a Kac model, as discussed above, with n positions and m scatterers. Now consider a subset n_s of those positions containing m_s scatterers. Let the members of the subset be contiguous, which implies a weakly interacting system because it is coupled to the rest of the Kac model only at the ends of the subset. The Kac model can now be thought of as the union of some number of these subsystems, where

$$n = \sum_s n_s \quad (14a)$$

$$m = \sum_s m_s \quad (14b)$$

An order parameter can be defined for each subsystem as

$$\Gamma_s(t) = \sum_{p=p_s+1}^{p_s+n_s} \eta_p(t) \quad (15)$$

where p_s indicates the starting point on the ring for the subsystem s . No loss of generality is incurred, however, by letting $p_s = 0$, because the order parameters are rotationally invariant.

4. NONEQUILIBRIUM PROPERTIES OF SUBSYSTEMS

As with the whole system, each Kac model subsystem must have a Poincaré cycle because it has a completely deterministic Hamiltonian.⁽⁸⁾ This Poincaré cycle time T can be no greater than that of the whole system

$[T = n, 2n \text{ for } (-1)^m = +1, -1]$ because at such a time the microstate of the system has returned to its initial condition. We will now demonstrate that the Poincaré cycle can be no smaller than this value.

Assume that there exists a time $T_s < n$ such that the microstate of the s th subsystem will be repeated for any arrangement of scatterers. From the equation of motion

$$\eta_j(T_s) = \epsilon_{j+1} \cdots \epsilon_{j+T_s} \eta_{j+T_s}(0) \quad (16)$$

we obtain

$$\epsilon_{j+1} \cdots \epsilon_{j+T_s} = 1 \quad (17)$$

for $1 \leq j \leq n_s$, which implies

$$\epsilon_j = \epsilon_{j+T_s} \quad (18)$$

This condition is inconsistent with the condition that the scatterers are placed randomly on the ring. In the thermodynamic limit the cases satisfying (18) have zero weight in the ensemble and can be disregarded. However, if $T_s = n$, Eq. (18) is satisfied for all members of the ensemble. Therefore the Poincaré cycle of a subsystem is the same as that of the whole system.

The relaxation time of a subsystem differs from that of the whole system. If the whole system is prepared in highly ordered state, $\Gamma(0) = 1$, all subsystems are also prepared in a highly ordered state, $\Gamma_s(0) = 1$. However, each subsystem will not necessarily approach equilibrium at the same rate. See Fig. 1. By analogy to Eq. (8) we write

$$\langle \Gamma_s(t) \rangle = \frac{n_s - 1}{n_s} (1 - 2\mu_s) \Gamma_s(t - 1) + \frac{1}{n_s} (1 - 2\mu) \Gamma(t - 1) \quad (19)$$

This can be iterated to obtain

$$\begin{aligned} \langle \Gamma_s(t) \rangle &= \left(\frac{n_s - 1}{n_s} \right)^t (1 - 2\mu_s)^t + \frac{1}{n_s} \sum_{k=0}^{t-1} (1 - 2\mu)^k (1 - 2\mu_s)^{t-k} \left(\frac{n_s - 1}{n_s} \right)^k \\ &= \left(\frac{n_s - 1}{n_s} \right)^t (1 - 2\mu_s)^t \frac{2(n_s - 1)(\mu_s - \mu)}{2n_s(\mu_s - \mu) + 1 - 2\mu_s} + \frac{(1 - 2\mu)^{t+1}}{2n_s(\mu_s - \mu) + 1 - 2\mu_s} \end{aligned} \quad (20)$$

where the prepared state is $\Gamma(0) = 1$.

To simplify Eq. (20), consider two cases. In the first case let the densities μ and μ_s be nearly identical,

$$\mu_s - \mu = O(1/n_s) \quad (21)$$

The first term in Eq. (20) is small compared with the second and we are left with

$$\langle \Gamma_s(t) \rangle \approx (1 - 2\mu)^{t+1} / (1 - 2\mu_s) \approx (1 - 2\mu_s)^t \quad (22)$$

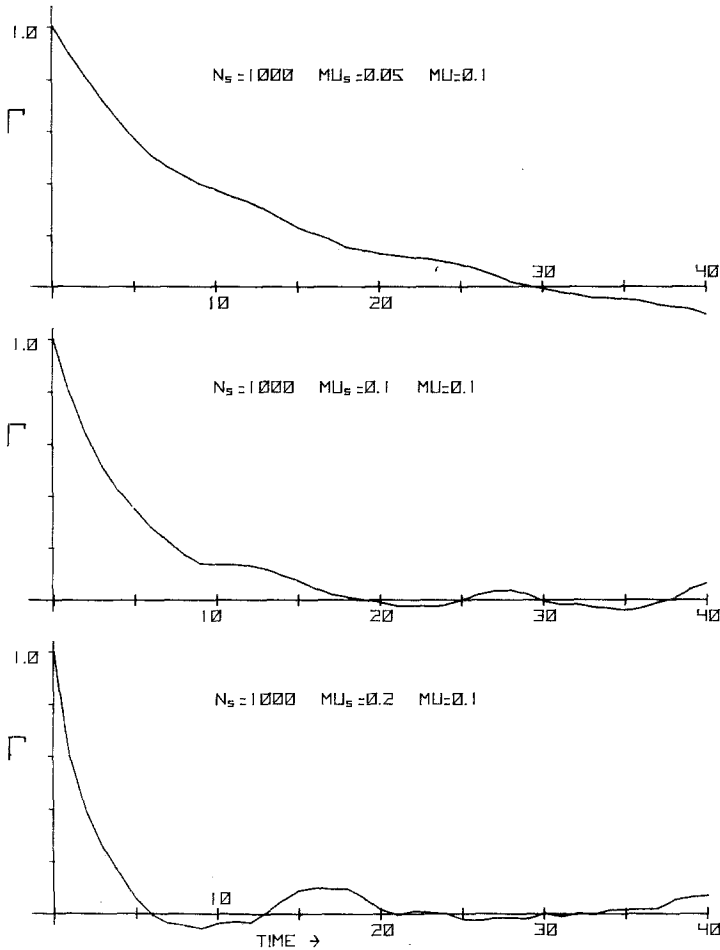


Fig. 1. Comparative relaxations of subsystems with different scatter densities.

as one expects in the case where the subsystem is similar to the whole system.

In the second case let the densities be significantly different, i.e.,

$$\mu_s - \mu = O(\mu) \tag{23}$$

Now the second term drops out of Eq. (20) and we are left with

$$\langle \Gamma_s(t) \rangle \approx [(n_s - 1)/n_s]^{t+1} (1 - 2\mu_s)^t \approx (1 - 2\mu_s)^t \tag{24}$$

and in both cases the relaxation time is

$$\tau_s \approx 1/2\mu_s \tag{25}$$

Table I. Chronology of the Kac Model

Time	Behavior of Kac model	
	Subsystems with similar densities	Subsystems with significantly different densities
0	Highly ordered	Highly ordered
$\ll \tau$	All systems are approaching equilibrium	All systems are approaching equilibrium
$\approx \tau$	All systems are approaching equilibrium	Most systems are approaching equilibrium, others are at equilibrium
$\gg \tau$	All systems are at equilibrium	Some systems are approaching equilibrium, others are at equilibrium
$\gg \tau$	All systems are at equilibrium	All systems are at equilibrium
$\ll n - \tau$	All systems are at equilibrium	Some systems are moving away from equilibrium, others are at equilibrium
$\approx n - \tau$	All systems are moving away from equilibrium	Most systems are moving away from equilibrium, others are at equilibrium
$\gg n - \tau$	All systems are moving away from equilibrium	All systems are moving away from equilibrium
$< n$	Highly ordered	Highly ordered

We are left with the rather remarkable result that the relaxation times of the subsystems are not dependent upon the relaxation time of the whole system in any direct way. This means that the whole system may be at equilibrium while some of its constituent subsystems are still approaching equilibrium. The result is less remarkable when one considers that for large n and n_s the probability that μ and μ_s will differ significantly is negligible for any subsystem which is a finite fraction of the whole system. However, if the restriction of random scatterer placement is relaxed to allow for density variation among the subsystems, then we have the situation where different subsystems reach equilibrium at different times.

The existence of a common Poincaré cycle and different relaxation times allows for the set of situations summarized in Table I. Note especially that there is no situation where a finite fraction of the subsystems are approaching equilibrium while another finite fraction are moving away from equilibrium. In the Kac model the arrow of time is preserved.

5. CONCLUSION

We have addressed ourselves to one of the two aspects of the arrow of time discussed in the introduction, "Why do all clocks go in the same direction?" In the Kac model all clocks (subsystems) go in the same direction

because they all have the same Poincaré cycle. This results from their mutual interaction, which has been restricted by design to be as weak as the dynamics of the Kac model will allow. The interaction is weak enough that it has no effect on the relaxation times of the subsystems. The clocks therefore appear to be independent of one another as they approach equilibrium, but the weak interaction is sufficient to assure that they all run in the same direction.

The foregoing implies (but, of course, does not prove) the following cosmological picture of a large, finite universe. An order parameter in this picture is neither a minimum nor a maximum at the temporal point we define as the present, but it has a well-defined, nonzero slope. The order parameters of all observable clocks have slopes with the same sign. The Kac model gives an indication of the reason for this phenomenon. There exists a time when the universe and all its subsystems are highly ordered. At any other time clocks must be moving in the same direction because even the weakest interaction between subsystems will give them the same Poincaré cycle. This explanation does not depend on whether the present is located on the slope to the right or to the left of the point of maximum order. It depends solely on the existence of a point when the universe is highly ordered. If such a point exists, then the behavior of the Kac model is consistent with the observed behavior of clocks.

APPENDIX. STEEPEST DESCENT CALCULATION OF $\langle \Gamma \rangle$

Equation (5) represents an exact expression for the ensemble average of the order parameter Γ . Any evaluation of this expression, however, is impractical because the number of terms in the expression is dependent upon the system size n . Therefore this expression is not appropriate for analyses requiring the thermodynamic limit. It is desirable to rewrite this expression in a form which is amenable to evaluation for large systems. Indeed, we would like an expression which simplifies as n increases rather than one which becomes more complex.

To accomplish this, we make use of a Laplace transform and its inverse to rewrite Eq. (5) in the form of an integral. Introducing the variable l conjugate to m , we have

$$\begin{aligned}
 \langle \Gamma \rangle &= \binom{n}{m}^{-1} \sum_s (-1)^s \binom{t}{s} \binom{n-t}{m-s} \\
 &= \binom{n}{m}^{-1} \frac{1}{2\pi i} \int_{-i\pi+\gamma}^{i\pi+\gamma} dl e^{lm} \sum_s (-1)^s \binom{t}{s} \sum_{m'} \binom{n-t}{m'-s} e^{-lm'} \\
 &= \left[2\pi i \binom{n}{m} \right]^{-1} \int_{-i\pi+\gamma}^{i\pi+\gamma} dl e^{lm} (1 + e^{-l})^{n-t} (1 - e^{-l})^t \tag{A.1}
 \end{aligned}$$

which is of the form

$$I = A \int_{-i\pi + \gamma}^{i\pi + \gamma} dl e^{ng(l)} \quad (\text{A.2})$$

where

$$g(l) = \frac{m}{n} l + \frac{n-t}{n} \ln(1 + e^{-l}) + \frac{t}{n} \ln(1 - e^{-l}) \quad (\text{A.3})$$

is of order unity, thereby implying that the integral is of the form which can be evaluated by the method of steepest descents.

We wish to distort the path of integration to follow a contour of constant phase. The phase in the case of this integral is necessarily equal to zero because the integrand is real at all points on the real axis, which must be crossed to perform the integration. No problems arise in the distortion of the contour because the integrand is an entire function.

We first attempt to see if there exist any paths of zero phase connecting the points $(\gamma, i\pi)$ and $(\gamma, -i\pi)$. The imaginary part of $g(l)$ can be written as

$$\text{Im } g(l) = \frac{m}{n} y - \frac{n-t}{n} \tan^{-1} \frac{\sin y}{e^x + \cos y} + \frac{t}{n} \tan^{-1} \frac{\sin y}{e^x - \cos y} \quad (\text{A.4})$$

where $l = x + iy$.

It is instructive to examine (A.4) for arbitrarily small values of y :

$$\text{Im } g(l) = \frac{m}{n} y - \frac{n-t}{n} \frac{y}{e^x + 1} + \frac{t}{n} \frac{y}{e^x - 1}, \quad x \neq 1 \quad (\text{A.5})$$

and

$$\text{Im } g(l) = \frac{m}{n} y - \frac{n-t}{n} \frac{y}{e^x + 1} + \frac{t}{n} \frac{\pi}{2}, \quad x = 1 \quad (\text{A.6})$$

Setting (A.5) and (A.6) equal to zero yields a real value for x only under the conditions

$$t \leq \frac{1}{2}n - [m(n-m)]^{1/2}, \quad t \geq \frac{1}{2}n + [m(n-m)]^{1/2}, \quad x \neq 1 \quad (\text{A.7})$$

and

$$t = 0, \quad x = 1 \quad (\text{A.8})$$

We conclude that conditions (A.7) are necessary to establish a path of zero phase which is a legitimate distortion of the integration contour. Numerical calculations for (A.4) demonstrate that these conditions are also sufficient.

The saddle points of the function $g(l)$ are determined by

$$\frac{dg}{dl} = \frac{m}{n} - \frac{n-t}{n} \frac{e^{-l}}{1 + e^{-l}} + \frac{t}{n} \frac{e^{-l}}{1 - e^{-l}} = 0 \quad (\text{A.9})$$

$$e^l = \frac{n - 2t \pm [(n - 2t)^2 - 4m(n - m)]^{1/2}}{2m} \quad (\text{A.10})$$

which has real roots for the values of t satisfying the conditions (A.7).

We know from Eq. (A.5) that the path of integration through the saddle point is perpendicular to the real axis. Therefore we differentiate twice with respect to y to determine which of the saddle points satisfies the steepest descent condition:

$$g''(l) = \frac{\partial^2 g}{\partial y^2} = -\frac{n-t}{n} \frac{e^t}{(1+e^t)^2} + \frac{t}{n} \frac{e^t}{(1-e^t)^2} < 0 \tag{A.11}$$

The roots satisfying condition (A.11) are

$$e^t = \frac{n-2t + [(n-2t)^2 - 4m(n-m)]^{1/2}}{2m}, \quad t \leq \frac{n}{2} - [m(n-m)]^{1/2} \tag{A.12}$$

$$e^t = \frac{n-2t - [(n-2t)^2 - 4m(n-m)]^{1/2}}{2m}, \quad t \geq \frac{n}{2} + [m(n-m)]^{1/2}$$

Numerical calculations show that the zero-phase path passing through each of these points completes a legitimate distortion of the original contour with $\gamma = 0$. The conditions for the steepest descent calculation are satisfied and

$$\langle \Gamma \rangle = \{ \exp[ng(l)] \} / \binom{n}{m} [-ng''(l)]^{1/2} \tag{A.13}$$

where $g(l)$ and $g''(l)$ are calculated using Eqs. (A.12), (A.3), and (A.11).

We have obtained an expression for $\langle \Gamma \rangle$ which is more amenable to calculation than is Eq. (5). The greater virtue of this expression, however, is that it is readily used to obtain approximate solutions to various orders of the quantities t/n and $(n-t)/n$. To demonstrate its usefulness, we expand the above expressions and evaluate them to the first power in t/n and $(n-t)/n$. Expanding about $t = 0$, we obtain from (A.13)

$$e^t = \frac{n-2t + (n-2m)\{1 - [4t(n-t)/(n-2m)]^{1/2}\}}{2m}$$

$$\approx \frac{n-m}{n} - \frac{\alpha t}{m}, \quad \alpha = 1 + \frac{n-t}{n-2m} \tag{A.14}$$

Therefore

$$e^{ng(l)} = e^{lm}(1+e^{-t})^{n-t}(1-e^{-t})^t$$

$$= e^{lm}(1+e^{-t})^n \tanh^t(l/2)$$

$$\approx \left(\frac{n-m-\alpha t}{m}\right)^m \left(\frac{n-\alpha t}{n-m-\alpha t}\right)^n \tanh^t \left\{ \frac{1}{2} \left[\ln \left(\frac{n-m-\alpha t}{m} - \frac{\alpha t}{m} \right) \right] \right\}$$

$$\approx \frac{n^n}{m^m(n-m)^{n-m}} \left(\frac{n-\alpha t}{n}\right)^n \left(\frac{n-m-t}{n-m}\right)^{-n+m}$$

$$\times \tanh^t \left\{ \frac{1}{2} \left[\ln \frac{n-m}{m} - \frac{\alpha t}{n-m} \right] \right\}$$

$$\begin{aligned}
&= \frac{n^n}{m^m(n-m)^{n-m}} \exp\left[n \ln\left(1 - \frac{\alpha t}{n}\right) - (n-m) \ln\left(1 - \frac{\alpha t}{n-m}\right)\right] \\
&\quad \times \tanh^t\left\{\frac{1}{2}\left[\ln \frac{n-m}{m} - \frac{\alpha t}{n-m}\right]\right\} \\
&\approx \frac{n^n}{m^m(n-m)^{n-m}} \exp\left[-\frac{\alpha^2 t^2}{2n} + \frac{\alpha^2 t^2}{2(n-m)}\right] \tanh^t\left\{\frac{1}{2}\left[\ln \frac{n-m}{m} - \frac{\alpha t}{n-m}\right]\right\} \\
&= \frac{n^n}{m^m(n-m)^{n-m}} \exp\left[\frac{\alpha^2 m t^2}{2n(n-m)}\right] \tanh^t\left\{\frac{1}{2}\left[\ln \frac{n-m}{m} - \frac{\alpha t}{n-m}\right]\right\} \quad (\text{A.15})
\end{aligned}$$

From (A.11), we have

$$\begin{aligned}
g''(l) &\approx -\frac{(n-t)m(n-m-\alpha t)}{n(n-\alpha t)^2} + \frac{tm(n-m-\alpha t)}{n(n-2m-\alpha t)^2} \\
&= -\frac{m(n-m-\alpha t)}{n} \left[\frac{(n-t)(n+\alpha t)^2}{(n^2-\alpha^2 t^2)^2} - \frac{t(n+2m+\alpha t)^2}{[n^2-(2m+\alpha t)^2]^2} \right] \\
&\approx -\frac{m(n-m-\alpha t)}{n^5} [(n-t)(n+\alpha t)^2 - t(n+2m+\alpha t)^2] \\
&\approx -\frac{m(n-m-\alpha t)}{n^5} [n^3 + 2(\alpha-1)tn^2] \\
&\approx -\frac{m}{n^2} [n-m+(\alpha-2)t] \quad (\text{A.16})
\end{aligned}$$

But

$$\alpha - 2 = \frac{n-t}{n-2m} - 1 = \frac{2m-t}{n-2m} \quad (\text{A.17})$$

yielding

$$g''(l) \approx -\frac{m(n-m)}{n^2} \left[1 + \frac{(2m-t)t}{(n-2m)(n-m)}\right] \approx -\frac{m(n-m)}{n^2} \quad (\text{A.18})$$

to the first power in t/n .

Therefore, with the help of Stirling's formula, Eq. (A.13) becomes

$$\langle \Gamma \rangle \approx \exp\left[\frac{\alpha^2 m t^2}{n(n-m)}\right] \tanh^t\left\{\frac{1}{2}\left[\ln \frac{n-m}{m} - \frac{\alpha t}{n-m}\right]\right\} \quad (\text{A.19})$$

A similar expansion of t about n yields

$$\langle \Gamma \rangle \approx (-)^m \exp\left[\frac{\beta^2 m(n-t)^2}{n(n-m)}\right] \tanh^{n-t}\left\{\frac{1}{2}\left[\ln \frac{n-m}{m} - \frac{\beta(n-t)}{n-m}\right]\right\} \quad (\text{A.20})$$

where $\beta = 1 + [t/(n-2m)]$.

We can obtain Eq. (11) by dropping the first-order correction terms in t . Equation (A.19) becomes

$$\langle \Gamma \rangle \approx \tanh^t \left(\frac{1}{2} \ln \frac{n-m}{m} \right) = \left\{ \frac{[(n-m)/m] - 1}{[(n-m)/m] + 1} \right\}^t = (1 - 2\mu)^t \quad (\text{A.21})$$

while Eq. (A.20) becomes

$$\langle \Gamma \rangle \approx (-1)^m (1 - 2\mu)^{n-t} \quad (\text{A.22})$$

The evaluation allows us to examine the nature of the exponential approximation. Numerical examples using Eqs. (A.19) and (A.21) demonstrate that the approach to equilibrium is actually somewhat more rapid than an exponential decay, following the exponential form closely at first and then dropping below it measurably just before the equilibrium region is reached. Exact calculations using Eq. (5) verify this in systems small enough to be conveniently analyzed by this method.

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